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1989 J. Phys. A: Math. Gen. 22 L865

(http://iopscience.iop.org/0305-4470/22/18/002)

J. Phys. A: Math. Gen. **22** (1989) L865–L870. Printed in the UK

LETTER TO THE EDITOR

# Conformal anomaly for the exactly integrable SU($N$) magnets

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Received 9 May 1989, in final form 19 June 1989

**Abstract.** The central charge for the integrable higher-spin SU($N$) magnets are calculated by solving numerically their associated nested Bethe ansatz equations.

The hypothesis that fluctuations at the critical point are conformally invariant (Polyakov 1970) has led to a remarkable progress in the two-dimensional arena of critical phenomena (see, e.g., Cardy 1987 for a review). As a consequence of this assumption, the universality classes are labelled by the central charge $c$, or conformal anomaly of the associated conformal algebra (Virasoro), whose irreducible representations determine the scaling dimensions of operators describing the critical behaviour (Belavin *et al* 1984a, b). Moreover, the requirement of reflection positivity (unitarity) of the transfer matrix (Friedan *et al* 1984) gives for $0<c<1$ a complete list of universality classes.

For conformal theories with $c>1$, unitarity is not enough to restrict the values of $c$. However, when the primary fields obey an algebra larger than Virasoro algebra, $c$ is again restricted to a countable set. In the case where these fields obey a Kac–Moody algebra, with topological charge $k$ $(1, 2, 3, \ldots)$ and an associated semi-simple group G, the conformal anomaly is given by (Knizhnik and Zamolodchikov 1984)

$$c=\frac{kD^{\mathrm{G}}}{k+C_A^{\mathrm{G}}} \tag{1}$$

where $D^{\mathrm{G}}$ is the dimension of the group G and $C_A^{\mathrm{G}}$ is the quadratic Casimir operator in the adjoint representation of G.

The conformal invariance of the infinite statistical mechanics systems, at the critical point, also give us important predictions (Cardy 1987) for the eigenspectrum of the associated transfer matrix or Hamiltonian, in the finite-size strip geometry. For a Hamiltonian with finite size $L$ and periodic boundary conditions, the leading behaviour as $L\to\infty$ of the ground-state energy per particle $E_0(L)$, at the critical point, is governed by the conformal anomaly $c$ (Blöte *et al* 1986, Affleck 1986):

$$E_0(L)=E_0(\infty)-\pi c\zeta_s/6L^2+\mathrm{o}(L^{-2}) \tag{2}$$

where $E_0(\infty)$ is the bulk limit of the ground-state energy per particle and $\zeta_s$ is the sound velocity, which can be inferred from the low-momentum behaviour of the energy–momentum dispersion relation of the model (Gehlen and Rittenberg 1986). The relation (2) give us a powerful way to calculate the conformal anomaly from the finite-size behaviour of the ground-state energy. From the fact that the finite-size effects of a two-dimensional statistical model in a strip of size $L$ are equivalent to the

0305-4470/89/180865+06$02.50  

finite-temperature effects of a one-dimensional quantum Hamiltonian with temperature $T$ ($L=\beta=1/T$) the conformal anomaly can also be estimated from the low-temperature behaviour of the specific heat of the infinite system.

On the other hand, with the development of the quantum inverse scattering method it was shown that associated with a $p$-representation ($p=1$ is the fundamental representation) of a semi-simple Lie group G it is possible to construct an exact integrable antiferromagnetic model (Ogievetsky and Wiegmann 1986, Ogievetsky *et al* 1987, Reshetikhin and Wiegmann 1987). In the case where $G=SU(2)$, the integrable model corresponding to the $p$-representation is the Heisenberg model with spin $S=P/2$ introduced by Babujian (1982) and Takhtajan (1982). Its conformal anomaly, which was calculated either from its low-temperature specific heat (Affeck 1986) or from the finite-size behaviour (2) of the ground-state energy (Alcaraz and Martins 1988a, b, Ziman and Schultz 1987), is given by

$$c=\frac{3S}{1+S}. \tag{3}$$

It follows from (1) that (3) corresponds to the conformal anomaly of the SU(2) Kac–Moody algebra with charge $k=2S$. The fundamental representation ($p=1$) of the $G=SU(n)$ model corresponds to the models introduced by Uimin (1970) and Sutherland (1975), while the model corresponding to higher-spin representations ($p=2,3,\ldots$) was derived by Andrei and Johannesson (1984), and Johannesson (1986).

For arbitrary semi-simple Lie group G these models are multicomponent generalisations of the standard XXX model (SU(2)) and the eigenenergies are obtained through the nested Bethe ansatz (NBA) with $r$ components, where $r$ is the rank of G. In the case of the $p$-representation of the $SU(N)$ group the energies are given in terms of the roots

$$\{\lambda_i^{(r)},\ i=1,2,\ldots,M^{(r)};\ r=1,2,\ldots,N-1\}$$

of the nested Bethe ansatz equations (Johannesson 1986):

$$\left\{\frac{\lambda_j^{(1)}+\mathrm{i}p/2}{\lambda_j^{(1)}-\mathrm{i}p/2}\right\}^L=\prod_{\substack{k=1\\k\neq j}}^{M^{(1)}}\left(\frac{\lambda_j^{(1)}-\lambda_k^{(1)}+\mathrm{i}}{\lambda_j^{(1)}-\lambda_k^{(1)}-\mathrm{i}}\right)\prod_{k=1}^{M^{(2)}}\left(\frac{\lambda_j^{(1)}-\lambda_k^{(2)}+\mathrm{i}/2}{\lambda_j^{(1)}-\lambda_k^{(2)}-\mathrm{i}/2}\right) \tag{4}$$

where

$$j=1,2,\ldots,M^{(1)}$$

and

$$\prod_{k=1}^{M^{(r)}}\left(\frac{\lambda_j^{(r)}-\lambda_k^{(r)}+\mathrm{i}}{\lambda_j^{(r)}-\lambda_k^{(r)}-\mathrm{i}}\right)=\prod_{k=1}^{M^{(r-1)}}\left(\frac{\lambda_j^{(r)}-\lambda_k^{(r-1)}+\mathrm{i}/2}{\lambda_j^{(r)}-\lambda_k^{(r)}-\mathrm{i}/2}\right)\prod_{k=1}^{M^{(r+1)}}\left(\frac{\lambda_j^{(r)}-\lambda_k^{(r+1)}+\mathrm{i}/2}{\lambda_j^{(r)}-\lambda_k^{(r+1)}-\mathrm{i}/2}\right) \tag{5}$$

where $j=1,2,\ldots,M^{(r)}$, $r=2,3,\ldots,N-1$, and $NL>M^{(1)}>M^{(2)}>\ldots>M^{(N-1)}$, $M^{(N)}=0$. The eigenenergies per site $E^{p,N}(L)$ are given only in terms of the roots $\{\lambda_1^{(1)}\}$:

$$E^{p,N}(L)=-\frac{p}{2}\sum_{j=1}^{M^{(1)}}\frac{1}{(\lambda_j^{(1)})^2+(p/2)^2}. \tag{6}$$

In the fundamental representation ($p=1$) the roots corresponding to the ground state are real and in this case by extending standard methods (de Vega and Woynarovich 1985, Hammer 1986, Woynarovich and Eckle 1987) the leading behaviour of the ground-state energy was obtained analytically (Pokrovskiĭ and Tsvelick 1987, de Vega 1988, Suzuki 1988) and using (2) the conformal anomaly is given by

$$c = \operatorname{rank} \mathrm{G}$$

which for the SU($N$) case gives

$$c = N - 1.$$

Comparing with (1), we see that they coincide with the conformal anomaly of a Kac-Moody algebra with associated group G and topological charge $k = 1$. For higher representations ($p > 1$) the roots representing the ground state are now complex and there is no analytical method to compute the large, but finite, behaviour of the eigenenergies.

In this letter we calculate the conformal anomaly of higher representations ($p > 1$) of the SU($N$) models by solving numerically the NBA (5). The NBA for the ground state of the $L$-sites Hamiltonian are given by considering in (5) $M^{(r)} = pL(N-r)/N$; $r = 1, 2, \ldots, N-1$, and there are $p(L/2)(N-1)$ coupled non-linear equations. Consequently, in order to estimate the conformal anomaly, by using (2), we should consider lattice sizes which are multiples of $N$. The bulk limit $E_0^{p,N}(\infty)$ of the ground-state energy per particle and the sound velocity are given by (Johannesson 1986)

$$E_0^{p,N}(\infty) = \frac{1}{N} \sum_{j=1}^{p} \left\{ \psi\left(\frac{j}{N}\right) - \psi\left(\frac{j+N-1}{N}\right) \right\} \tag{7}$$

and

$$\zeta_s^N = \pi / N \tag{8}$$

respectively. In (7) $\psi(x)$ is the Euler psi function. The results (7) and (8) are obtained by using the string assumption to convert the complex equations (5) into real ones. This assumption asserts that in the bulk limit the roots corresponding to the ground state $\{\lambda_j^{(r)}; j = mp + \alpha;\ \alpha = 1, 2, \ldots, p;\ m = 0, 1, \ldots, L(N-r)/N - 1\}$, for $r = 1, 2, \ldots, N-1$, cluster together forming a sea of $L(N-r)/N$ strings of size $p$

$$\{\lambda_{m,\alpha}^{(r)} = \Delta_m^{(r)} + \mathrm{i}(p+1-2\alpha)/2;\ \alpha = 1, 2, \ldots, p;\ m = 1, \ldots, L(N-r)/N\}$$

where $\Delta_j^{(r)}$ are real numbers.

For large lattices, instead of solving (5) directly we first solve them by using the string assumption and then we use these roots as an initial guess for the solution of (5). In table 1 we show the ground-state energy for some values of $L$, $N$ and $p$, together with their bulk limit results obtained from (7). Using equations (2) and (8), the conformal anomaly can be estimated by extrapolating the sequence

$$c_p^N(L) = (E_0^{p,N}(\infty) - E_0(L)) 6L^2 / \pi\zeta_s. \tag{9}$$

In table 2 we show some of our estimates for some representation of SU(3) and SU(4). Our results clearly indicate the conformal anomaly

$$c = \frac{p(N^2-1)}{p+N} \tag{10}$$

for the $p$-representation of the SU($N$) model. Since for $\mathrm{G} = \mathrm{SU}(N)$, $\dim(\mathrm{G}) = N^2 - 1$ and $C_A^{\mathrm{G}} = N$, the central charge (10) is the same as that of the SU($N$) Kac-Moody algebra with topological charge $p$. As we have already discussed, the above result can also be derived from the low-temperature behaviour of the specific heat of these models

**Table 1.** Ground-state energy per particle, for the SU($N$) exactly integrable model in its $p$-representation ($p = 1, 2, 3$) in a ring of $L$ sites ($a$) SU(3) ($b$) SU(4). The infinite-size limits are given by (7).

| $L$ | $p = 1$ | $p = 2$ | $p = 3$ |
|---|---|---|---|
| ($a$) SU(3) | | | |
| 6 | −0.883 7959 | −1.299 6589 | −1.567 0639 |
| 18 | −0.855 0231 | −1.252 5034 | −1.506 9052 |
| 30 | −0.852 8305 | −1.248 9718 | −1.502 4642 |
| 42 | −0.852 2298 | −1.248 0067 | −1.501 2534 |
| 48 | −0.852 0834 | −1.247 7717 | −1.500 9588 |
| 54 | −0.851 9831 | −1.247 6107 | −1.500 7570 |
| 60 | −0.851 9114 | −1.247 4957 | −1.500 6129 |
| $\infty$ | −0.851 6060 | −1.247 0063 | −1.5 |
| ($b$) SU(4) | | | |
| 8 | −0.932 5631 | −1.380 5200 | −1.671 5006 |
| 24 | −0.914 7159 | −1.350 1834 | −1.631 8209 |
| 32 | −0.913 7699 | −1.348 5975 | −1.629 7711 |
| 40 | −0.913 3333 | −1.347 8666 | −1.628 8278 |
| 48 | −0.913 0965 | −1.347 4706 | −1.628 3169 |
| 56 | −0.912 9538 | −1.347 2321 | −1.628 0095 |
| 64 | −0.912 8613 | −1.347 0775 | −1.627 8103 |
| $\infty$ | −0.912 5595 | −1.346 5736 | −1.627 1613 |

(Pokrovskiĭ and Tsvelik 1987). Our results only relate the conformal anomaly of the SU($N$) spin models with that of a SU($N$) Kac–Moody algebra. However, these results together with the previous analysis of the SU(2) spin models, with arbitrary representations $p$ (Affeck 1986, Alcaraz and Martins 1988a, b) and of the fundamental representation of an arbitrary semi-simple group G (Pokrovskiĭ and Tsvelick 1987, de Vega 1988, Suzuki 1988) indicate the conjecture that the Wess–Zumino–Witten–Novikov model (Knizhnik and Zamolodchikov 1984) with associated group G and topological charge $p$ is the underlying field theory describing the critical fluctuations of these exactly integrable models.

Our numerical results also indicate, in a similar fashion as in the SU(2) model (Alcaraz and Martins 1988a, b) the appearance of logarithmic corrections in (2). These corrections indicate, as in the SU(2) model (Alcaraz and Martins 1988a, b), the occurrence of a marginal operator governing the finite-size corrections (Cardy 1986).

As a final remark it is interesting to observe that the conformal anomaly (10) can be decomposed into a sum of $(N-1)$ conformal anomalies of free-field theories ($c = 1$) and the conformal anomaly of a SU($N$) generalised parafermionic theory $c = (N-1)[p(N+1)/(p+N)-1]$, introduced by Gepner (1987). This fact indicates that, as in the SU(2) model (di Francesco *et al* 1988, Alcaraz and Martins 1988c, 1989) the partition function of the SU($N$) exactly integrable models can be expressed by the sum of the partition function of Coulomb gases and generalised parafermionic theories. The calculation of the whole operator content of these models we leave for a future work.

This work was supported in part by CNPq Brasil and Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP).

**Table 2.** Finite-size sequences of the quantities $C_N^p(L)$, $p = 1, 2, 3$ for the (*a*) SU(3) and (*b*) SU(4) models. We also present the extrapolated results together with their conjectured values obtained from (1).

| $L$ | $p=1$ | $p=2$ | $p=3$ |
|---|---|---|---|
| (*a*) SU(3) | | | |
| 6 | 2.113 464 | 3.456 968 | 4.403 156 |
| 18 | 2.019 137 | 3.248 305 | 4.080 305 |
| 30 | 2.009 793 | 3.226 209 | 4.044 798 |
| 42 | 2.006 700 | 3.218 590 | 4.032 286 |
| 48 | 2.005 840 | 3.216 421 | 4.028 681 |
| 54 | 2.005 204 | 3.214 798 | 4.025 969 |
| 60 | 2.004 716 | 3.213 538 | 4.023 852 |
| Extrapolated | 1.99 (9) | 3.20 (1) | 4.00 (1) |
| Conjectured | 2.0 | 3.2 | 4.0 |
| (*b*) SU(4) | | | |
| 8 | 3.113 154 | 5.283 055 | 6.900 490 |
| 24 | 3.020 394 | 5.056 114 | 6.526 563 |
| 32 | 3.013 988 | 5.039 572 | 6.498 565 |
| 40 | 3.010 688 | 5.030 877 | 6.483 703 |
| 48 | 3.008 711 | 5.025 579 | 6.474 577 |
| 56 | 3.007 404 | 5.022 030 | 6.468 424 |
| 64 | 3.006 481 | 5.019 491 | 6.463 996 |
| Extrapolated | 3.00 (0) | 5.00 (1) | 6.4 (3) |
| Conjectured | 3.0 | 5.0 | 6.4285 |

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